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COMMENT

2D directed compact site animals on a lattice of finite width

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Abstract. We present a transfer matrix method for calculating the properties of directed compact site animals in two dimensions. The non-degeneracy of the largest eigenvalue of a truncated 50×50 matrix implies that the critical exponent $\theta = 0$. We also predict that the exponents $\nu_{\perp} = \frac{1}{2}$ and $\nu_{\|} = 1$.

In the past few years systems with directionality-dependent critical behaviour have been the focus of much attention. The most studied systems are directed percolation (Kinzel 1983 and references therein) and directed animals (Redner and Yang 1982, Dhar *et al* 1982). Recently we have proposed and studied the problem of directed compact site animals in two dimensions (see Bhat *et al* 1986).

In this comment we study the problem of directed compact site animals on a strip of finite width using transfer matrices. We begin by briefly reviewing the formalism used in Bhat *et al* (1986). A directed compact site animal is defined as a connected cluster of sites having no holes, with the constraint that all the sites in the cluster are reachable from a particular site (the origin) via a path that never goes opposite to the preferred direction. Thus all the sites enclosed by a directed compact animal are occupied which obviously is not true in the case of a simply directed animal.

We consider directed compact site animals on a strip of finite width m of a square lattice. The preferred direction lies along a diagonal of the lattice. The strip is infinite along this preferred direction and has periodic boundary conditions in the perpendicular direction.

The animal generating function is defined as the sum of the weights of all animals, the weight of an animal of size n being x^n,

$$F(x) = \sum_{n=0}^{\alpha} F_n x^n \tag{1}$$

where F_n is the number of animals of size n. For large n, F_n is expected to have the asymptotic form:

$$F_n \sim \mu^n \bar{n}^{\theta} \tag{2}$$

where μ is a lattice-dependent growth parameter or inverse critical fugacity and θ is a critical exponent. The generating function exhibits a power law singularity of the form

$$F(x) \sim |x_c - x|^{\theta - 1} \tag{3}$$

as $x \to x_c$. We can easily write down the relations between animals with different source sets as

$$F^1(x) = 1 + 2xF^1(x) + x^2F^{11}(x)$$
$$F^{11}(x) = 1 + 3xF^1(x) + 2x^2F^{11}(x) + x^3F^{111}(x)$$
$$F^{111}(x) = 1 + 4xF^1(x) + 3x^2F^{11}(x) + 2x^3F^{111}(x) + x^4F^{1111}(x) \tag{4}$$

and so on. This set of coupled recursions can be written in a closed form as

$$F(x, y) = -xF^1(x) + \frac{y}{(1-y)} + \frac{F(x, xy)}{y(1-y)^2} \tag{5}$$

where

$$F(x, y) = \sum_{L=1}^{\alpha} y^L F^L(x)$$

and

$$F(x, xy) = \sum_{L=1}^{\alpha} (xy)^L F^L(x). \tag{6}$$

Comparing the coefficients of a Taylor series expansion of $F(x, y)$ around $y = 0$ with those of $F(x, y)$ in equation (6) we get:

$$F^1(x) = \left.\frac{\partial F(x, y)}{\partial y}\right|_{y=0}$$
$$F^{11}(x) = \frac{1}{2!}\left.\frac{\partial F(x, y)}{\partial y}\right|_{y=0}. \tag{7}$$

From equation (7) one can generate all the recursion relations of set (4) (see Bhat *et al* 1986). The recursions of set (4) have obviously a matrix form and can be written as:

$$\begin{bmatrix} F^1(x) \\ F^{11}(x) \\ \vdots \\ F^m(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} 2x & x^2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 3x & 2x^2 & x^3 & 0 & 0 & \dots & 0 & 0 \\ 4x & 3x^2 & 2x^3 & x^4 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\ mx & (m-1)x^2 & & & & \dots & 2x^{m-1} & x^m \\ (m+1)x & mx^2 & & & & \dots & 3x^{m-1} & x^m \end{bmatrix} \begin{bmatrix} F^1(x) \\ F^{11}(x) \\ \vdots \\ F^m(x) \end{bmatrix}$$

or

$$\begin{bmatrix} F^1(x) \\ F^{11}(x) \\ \vdots \\ F^m(x) \end{bmatrix} = (1 - T(x))^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{8}$$

where

$$T(x) = \begin{bmatrix} 2x & x^2 & 0 & 0 & \dots & 0 \\ 3x & 2x^2 & x^3 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ (m+1)x & mx^2 & & \dots & & x^m \end{bmatrix}$$

truncated at $T_{m,m}$ is the transfer matrix. A closed form expression for the matrix $T(x)$ is written as

$$T_{ij}(x) = (i-j+2)x^j \qquad \text{for } j \leq i$$
$$= x^{i+1}\delta_{j,i+1} \qquad \text{for } j > i. \tag{9}$$

The largest eigenvalue $\lambda_m(x)$ corresponding to a strip of width m can be calculated numerically. We expect that the largest eigenvalue of $T(x)$ is non-degenerate when $x = x_c$ and is analytic at that point. This conclusion follows from the usual argument based on the theorem of Perron–Frobenius, which states that the largest eigenvalue of a finite matrix with positive elements is non-degenerate (Gantmacher 1960).

We diagonalised a truncated $T(x)$ matrix of size 50×50 for $x_c = 0.375\,67$ (Bhat *et al* 1986) and found that the largest eigenvalue is non-degenerate. The magnitude of the largest eigenvalue is 0.999 974 12 and the next largest eigenvalue has magnitude equal to 0.163 245 34 (see table 1). For the matrix of size 51×51 the largest eigenvalue for fixed x agrees with that of the 50×50 matrix up to eight decimal places and the gap between eigenvalues does not decrease significantly (see table 2).

We repeated the diagonalisation of matrix $T(x)$ at two different values of x near x_c and the results show that the largest eigenvalue is non-degenerate. The eigenvalues corresponding to different values of x are listed in table 1. It took us about six minutes of CPU time on an ICL 2960 computer. In the asymptotic limit the eigenvalue of maximum magnitude dominates and we can write

$$F_n \sim \lambda_1^n. \tag{10}$$

We have neglected the terms of the form $(\lambda_i/\lambda_1)^n$ in comparison with unity for $i = 2, 3, 4, \ldots$. By looking at equations (2) and (10) one concludes that the critical exponent $\theta = 0$, which is in agreement with our earlier results (Bhat *et al* 1986).

We now turn to the estimation of correlation length exponents $\nu_\perp$ and $\nu_\parallel$. As the number of sites n diverges the compact animals become anisotropic in shape. Thus for longitudinal cluster radius and transverse cluster radius respectively, we write

$$\xi_\perp \sim n^{\nu_\perp} \qquad \xi_\parallel \sim n^{\nu_\parallel}. \tag{11}$$

Table 1. The largest and second-largest eigenvalues for different values of x, for the 50×50 truncated matrix.

x	λ_1	λ_2
0.32	0.814 682 05	0.106 675 51
0.375 67	0.999 974 12	0.163 245 34
0.40	1.086 168 0	0.193 663 89

Table 2. The largest and second-largest eigenvalues for different values of x, for the 51×51 truncated matrix.

x	λ_1	λ_2
0.32	0.814 682 05	0.106 675 51
0.375 67	0.999 974 12	0.163 245 34
0.40	1.086 168 0	0.193 663 89

We observe that each directed step between the sites in the cluster makes a projection of $+2^{-1/2}$ on the (1, 1) diagonal and a projection of $\pm 2^{-1/2}$ perpendicular to the diagonal. Therefore $\xi_{\parallel} = \langle R_{\parallel n} \rangle$ equals $n/2^{1/2}$ while $\xi_{\perp} = \langle R_{\perp n}^2 \rangle^{1/2}$ equals $(n/2)^{1/2}$. From these relations and equation (11) we conclude that $\nu_{\parallel} = 1$ and $\nu_{\perp} = \frac{1}{2}$.

In summary, we obtain the exponents $\theta = 0$, $\nu_{\perp} = 0.5$ and $\nu_{\parallel} = 1$ for two-dimensional directed compact animals. We feel that the critical behaviour of directed compact animals might belong to the universality class of directed self-avoiding walks.

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